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1980 J. Phys. A: Math. Gen. 13 L295

(http://iopscience.iop.org/0305-4470/13/9/003)

J. Phys. A: Math. Gen. 13 (1980) L295–L298. Printed in Great Britain

LETTER TO THE EDITOR

Removing cut-offs from one-dimensional Schrödinger operators

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Received 13 June 1980

Abstract. This Letter deals with the following type of question. Suppose $V_0(x)$ is so singular near 0 that the quadratic form of $-\mathrm{d}^2/\mathrm{d}x^2 + V_0(x)$ is unbounded from below, but that $(-\mathrm{d}^2/\mathrm{d}x^2)_\mathrm{D} + V_0(x)$ is bounded below where D refers to Dirichlet boundary conditions (BCS) at 0. Let $H(a) = -\mathrm{d}^2/\mathrm{d}x^2 + V_a(x)$ where $V_a(x) \to V_0(x)$ pointwise (AE) as $a \downarrow 0$. Under some additional assumptions on V_a, we prove that $H(a) \to H_\mathrm{D}(0)$ in the norm resolvent sense. Typical for applications is the case where V_a are cut-off potentials or regularised potentials of some sort.

This Letter was inspired by a recent work of Gesztesy (1980, to be referred to as I) where it was shown that on $L^2(\mathbb{R})$ the operator

$$H(a) = -\mathrm{d}^2/\mathrm{d}x^2 - c/(|x| + a), \qquad c > 0, a > 0, \tag{1}$$

tends to

$$H_\mathrm{D}(0) = (-\mathrm{d}^2/\mathrm{d}x^2)_\mathrm{D} - c/|x| \tag{2}$$

in the strong resolvent sense, as $a \downarrow 0$. The subscript D in (2) and throughout this Letter means that we have a Dirichlet BC at 0. Following I, the behaviour of the bound states in the limit of small a is as follows: the ground state $E_0(a)$ of (1) tends to $-\infty$ whereas the higher energies approach the eigenvalues of $H_\mathrm{D}(0)$ monotonically from above. If $E_1(a) < E_2(a) < \ldots$ denote the excited states of $H(a)$, then $E_1(a)$, which has odd parity and is also the ground state of $H_\mathrm{D}(a)$, tends to $-c^2/4$. But also $E_2(a)$ has to converge to $-c^2/4$, for $H_\mathrm{D}(0)$ has doubly degenerate eigenvalues and the non-crossing rule prevents $E_2(a)$ from ever getting smaller than $-c^2/4$. Thus removing the cut-off from (1) means putting Dirichlet BCS at 0.

It is the object of this Letter to prove that some of the essential features of the results in I carry over to a wide class of potentials and various types of cut-offs. We shall strengthen the results of I by proving *norm* resolvent convergence instead of strong resolvent convergence (theorem 1). We study the Hamiltonian

$$H(a) = -\mathrm{d}^2/\mathrm{d}x^2 + V_a(x) \tag{3}$$

in the limit $a \downarrow 0$ where a denotes a parameter taking on values in some interval $0 \leqslant a \leqslant a_0$.

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We require:

(i) $$|V_a(x)| \leq c(1+|x|^{\gamma-\epsilon})/|x|^{\gamma}, \qquad a \in [0, a_0], \tag{4}$$

where $\epsilon > 0$, $0 < \gamma < 2$ and c is a constant;

(ii) $$V_a(x) \in L_1^{\text{loc}}, \qquad a \in (0, a_0], \tag{5}$$

(iii) $$V_a(x) \to V_0(x) \tag{6}$$

pointwise AE as $a \downarrow 0$;

(iv) $$\int_{-b}^{b} V_a(x)\,\mathrm{d}x \to -\infty \qquad \text{as } a \downarrow 0, \text{ for some } b > 0; \tag{7}$$

(v) $$\sup_{a \in (0,a_0)} \frac{\int_{-b}^{b} |V_a|\,\mathrm{d}x}{|\int_{-b}^{b} V_a(x)\,\mathrm{d}x|} \equiv \tau < \infty. \tag{8}$$

Our main theorem states the following.

Theorem 1. Let $H(a)$ be as in (3) and let $V_a(x)$ obey conditions (i)–(v). Then $H(a) \to H_{\mathrm{D}}(0)$ in the norm resolvent sense.

Before proving theorem 1 we need some preparations. We define the Birman–Schwinger kernel for $H(a)$ by

$$K_{\alpha,a}(x, y) = |V_a(x)|^{1/2} \frac{\mathrm{e}^{-\alpha|x-y|}}{2\alpha} (V_a(y))^{1/2} \tag{9}$$

where $V_a^{1/2}$ stands for $|V_a|^{1/2}\,\mathrm{sgn}\, V_a$ and $(2\alpha)^{-1} \exp(-\alpha|x-y|)$ is the kernel of the inverse of $-\mathrm{d}^2/\mathrm{d}x^2 + \alpha^2$. The corresponding kernel for Dirichlet BCS is

$$K_{\alpha,a;\mathrm{D}}(x, y) = |V_a(x)|^{1/2} \frac{\mathrm{e}^{-\alpha|x-y|} - \mathrm{e}^{-\alpha|x|}\,\mathrm{e}^{-\alpha|y|}}{2\alpha} (V_a(y))^{1/2}. \tag{10}$$

Then

$$K_{\alpha,a} = K_{\alpha,a;\mathrm{D}} + L_{\alpha,a} \tag{11}$$

where $L_{\alpha,a}$ is rank one with kernel

$$L_{\alpha,a}(x, y) = |V_a(x)|^{1/2} \frac{\mathrm{e}^{-\alpha|x|}\,\mathrm{e}^{-\alpha|y|}}{2\alpha} (V_a(y))^{1/2}. \tag{12}$$

As $a \downarrow 0$,

$$K_{\alpha,a;\mathrm{D}} \to K_{\alpha,0;\mathrm{D}} \tag{13}$$

in *norm*, since if $W(x)$ denotes the RHS of (4) we have that

$$|V_a(x)|^{1/2} |W(x)|^{-1/2} \to |V_0(x)|^{1/2} |W(x)|^{-1/2}$$

strongly. So convergence in norm follows from the *compactness* of the kernel (10) with V_a replaced by W. Also $\|K_{\alpha,a;\mathrm{D}}\| \to 0$ as $\alpha \to \infty$ uniformly in a. We also note that for any rank-one operator $A = \tilde{\psi}(\psi, \;)$

$$(A - z)^{-1} = -\frac{1}{z} + \frac{\tilde{\psi}(\psi, \;)}{z((\psi, \tilde{\psi}) - z)}. \tag{14}$$

With these tools the proof of theorem 1 is now merely a somewhat tedious application of the resolvent expansion for $(H(a)-E)^{-1}$.

Proof of theorem 1. Write R (resp. R_0) for $(H(a)+\alpha^2)^{-1}$ (resp. $(-d^2/dx^2+\alpha^2)^{-1}$), K_D for $K_{\alpha,a;D}$, L for $L_{\alpha,a}$ and put $P=\phi(\phi,\cdot)$ where $\phi=\exp(-\alpha|x|)/(2\alpha)^{1/2}$ and $R_D=R_0-P$. Pick α such that $\|K_D\|<1$ and $\tau\|K_D\|(1-\|K_D\|)^{-1}<1$ for $a\in[0,a_0]$. Then

$$R=R_D+P-(R_D+P)V_a^{1/2}(1+K_D+L)^{-1}|V_a|^{1/2}(R_D+P). \tag{15}$$

Now, using (14)

$$\begin{aligned}(1+K_D+L)^{-1}&=[1+(1+K_D)^{-1}L]^{-1}(1+K_D)^{-1}\\ &=\left(1-\frac{1}{1+c}[(1+K_D)^{-1}|V_a|^{1/2}\phi](V_a^{1/2}\phi,\cdot)\right)(1+K_D)^{-1}\end{aligned} \tag{16}$$

where $c=(\phi, V_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}\phi)$. So

$$\begin{aligned}R=R_D+P&-R_DV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}R_D-PV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}R_D\\
&-R_DV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}P-PV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}P\\
&+\frac{1}{1+c}[R_DV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}\phi](V_a^{1/2}\phi,\cdot)(1+K_D)^{-1}|V_a|^{1/2}R_D\\
&+\frac{1}{1+c}[R_DV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}\phi](V_a^{1/2}\phi,\cdot)(1+K_D)^{-1}|V_a|^{1/2}P\\
&+\frac{1}{1+c}[PV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}\phi](V_a^{1/2}\phi,\cdot)(1+K_D)^{-1}|V_a|^{1/2}R_D\\
&+\frac{1}{1+c}[PV_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}\phi](V_a^{1/2}\phi,\cdot)(1+K_D)^{-1}|V_a|^{1/2}P.\end{aligned} \tag{17}$$

First we lump together the terms of the form $P\ldots P$. This gives

$$\left(1-c+\frac{c^2}{1+c}\right)P=\frac{1}{1+c}P. \tag{18}$$

As $a\downarrow 0$, this term converges to zero in norm, since it follows from (5)–(8) and expanding $(1+K_D)^{-1}$ that

$$c\leqslant(\phi, V_a\phi)[1-\tau\|K_D\|(1-\|K_D\|)^{-1}]$$

where $(\phi, V_a\phi)\to-\infty$ on account of (7).

Next we collect terms of the form $P\ldots R_D$. They add up to

$$-\frac{\phi}{1+c}(\phi, V_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}R_D\,\cdot). \tag{19}$$

The norm of this operator is bounded by

$$\frac{\text{constant}}{|1+c|}\|V_a^{1/2}\phi\| \tag{20}$$

since $\||V_a|^{1/2}R_D\|$ is uniformly bounded in a. By (7) and (8), this term goes to zero also. The $R_D\ldots P$ terms can be handled in the same way.

On writing $(1+K_D)^{-1}=1-(1+K_D)^{-1}K_D$, a simple estimate shows that the $R_D \dots R_D$ term which involves ϕ is bounded by

$$\frac{\text{constant}}{|1+c|}(\|R_D V_a\phi\|^2+\|K_D|V_a|^{1/2}\phi\|^2). \tag{21}$$

Now

$$\|R_D V_a\phi\| \leq \|R_D|x|^{-\beta}\| \||x|^{\beta} V_a\phi\| \tag{22}$$

where $\beta=\frac{3}{2}-\delta$, $0<\delta<2-\gamma$. By assumption (4) the second factor on the RHS of (22) stays bounded as $a\downarrow 0$. Thus the first term in (21) tends to zero.

Similarly, choosing $\delta<2-\gamma$ and $\delta<1$, we have

$$\|K_D|V_a|^{1/2}\phi\|^2 \leq \|K_D|x|^{-\delta/2}\|^2\| |x|^{\delta/2}|V_a|^{1/2}\phi\|^2 \tag{23}$$

and

$$\begin{aligned}\| |x|^{\delta/2}|V_a|^{1/2}\phi\|^2 &= (|V_a|^{1/2}\phi, |x|^{\delta}|V_a|^{1/2}\phi)\\ &\leq (|V_a|^{1/2}\phi, |x||V_a|^{1/2}\phi)^{\delta}(\phi, |V_a|\phi)^{1-\delta}\end{aligned} \tag{24}$$

by Jensen's inequality with respect to the operator $|x|^{\delta}$ and its spectral measure. As $a\downarrow 0$, the first factor in (24) stays bounded, while the second diverges less rapidly than $(\phi, V_a\phi)$ on account of (7) and (8). So (21) tends to zero. Finally, using (13), we conclude that the remaining term

$$R_D - R_D V_a^{1/2}(1+K_D)^{-1}|V_a|^{1/2}R_D \tag{25}$$

converges in norm to the inverse of $(-d^2/dx^2)_D+V_0+\alpha^2$. This proves theorem 1.

Remarks

(1) Assumptions (i)–(v) include some cases of physical interest, for example (1) or cut-offs of the form $V_a(x)=\min[-V_0(x), 1/a]$, $V_0\leq 0$. Moreover, the following situation is incorporated as well. Suppose $V_0(x)=0$ and $V_a(x)=-1/a^{1+\delta}$ $(V_a(x)=0)$ when $|x|<a/2$ $(|x|>a/2)$, $0<\delta<1$.

(2) As in the Coulomb case (1), only the ground state of $H(a)$ tends to $-\infty$ as $a\downarrow 0$. That the ground state does, follows from (iv), noting that $(\varphi, H(a)\varphi)\to-\infty$ $(a\downarrow 0)$ if $\varphi\in C_0^{\infty}(R)$, $\varphi\equiv 1$ on $[-b, b]$. That the ground state is the only divergent eigenvalue can be seen as follows. Suppose $E_0(a)$, $E_1(a)$ would both tend to $-\infty$ as $a\downarrow 0$. Choose a normalised linear combination ψ of the corresponding eigenfunctions which satisfies $\psi(0)=0$. Then $\psi\in D(H_D(a))$ and $(\psi, H_D(a)\psi)=(\psi, H(a)\psi)\in[E_0(a), E_1(a)]$, contradicting the fact that $H_D(a)$ is bounded from below uniformly in a (on account of (i)). Alternatively, one can also argue by means of the Birman–Schwinger principle, which says that $-\alpha^2$ is an eigenvalue of $H(a)$ if and only if $-K_{\alpha,a}$ has eigenvalue 1. If α is large $\|K_{\alpha,a;D}\|$ is small, so that $L_{\alpha,a}$ is responsible for the ground state diverging. In leading order, $I_{\alpha,a}$ has eigenvalue -1 if

$$\int_{-\infty}^{\infty}\frac{V_a(x)}{2\alpha}e^{-2\alpha|x|}\,dx=-1, \tag{26}$$

which gives an implicit equation for $\alpha(a)$. For problem (1) one easily verifies that $\alpha^2\approx -c^2(\ln a)^2$, which is in agreement with I.

Reference

Gesztesy F 1980 *J. Phys. A: Math. Gen.* **13** 867